Aware that, by a further transformation, a much more convenient expression applicable to single and joint lives could be got, Mr. Jones gives this in addition; but it would be absurd to suppose that in doing so he was ignorant of the fact that it did not apply to all cases. As a proof of this, he has, in his recapitulation of formulæ, p. 215, only referred to single and joint lives.

Baily, in a Note at page 147 of his work, says, "Since $(\mathrm{ABC})^d$ is, by Prob. I., cor. 3, equal to $\mathrm{A}^o\mathrm{B}^o\mathrm{C}^o \times \frac{\alpha\beta\gamma}{abc}(1+\rho)^{-n}$, it is obvious that the present value, in the case of single or joint lives, might be more conveniently expressed by $s \times \frac{1-\rho\mathrm{A}^o.\mathrm{B}^o.\mathrm{C}^o}{1+\rho} \times \frac{\alpha\beta\gamma}{abc}(1+\rho)^{-n}$; and it is from this formula that I have deduced the rule in Question 28, Chap. XII. But that rule will not extend to *all* cases."

If any blame at all attaches to Jones, it can only be for his omitting to state that the formula $r^t p_{\overline{(m,\, m_1,\, m_2,\, \&c.)}\, t}^{\;v} \mathrm{A}_{\overline{(m+t,\, m_1+t,\, \&c.)}}^{\;v}$ did not apply to all cases.

I am, Sir,

Your obedient servant,

COLIN McCUAIG,

*Assistant Actuary Scottish Union Insurance Company.*

*Edinburgh*, 47, *George Street*,
3*rd December*, 1866.

# INSTITUTE OF ACTUARIES.

## PROCEEDINGS OF THE INSTITUTE.

*First Ordinary Meeting. Session* 1866–7.—*Monday*, 26*th November*, 1866.

The President in the Chair.

Read and confirmed the minutes of the anniversary meeting.
The following gentlemen were elected Associates, viz.:—

Alexander Gordon Brown.
William Braid.
Frederick J. Hallow.
Henry F. W. Cowley.

Mr. T. B. Sprague, M.A., read a paper "On the value of annuities payable half-yearly and quarterly."

Thanks were voted to Mr. Sprague, and the meeting adjourned to Monday, the 17th December.

*Second Ordinary Meeting. Session* 1866–7.—*Monday*, 17*th December*, 1866.

The President in the Chair.

Read and confirmed the minutes of the last ordinary meeting.
The following gentlemen were elected members of the Institute, viz.:—

*Fellow.*
Henry Gamble Hobson.

*Associates.*

| | |
|---|---|
| Augustus H. Browne. | Nicholas Hanhart. |
| Bernard Woods. | Leycester H. Greaves. |

A communication from Mr. Meikle, "On the arrangement of the data of Life Assurance Offices," and a paper by Mr. Sprague, "On the limitation of risks" (Part II.), were read.

Thanks were voted to Mr. Meikle and Mr. Sprague, and the meeting adjourned to the 28th January, 1867.

## SOLUTIONS OF THE SECOND YEAR'S EXAMINATION QUESTIONS.

In October, 1861, we published solutions by Mr. Sprague of the questions proposed at the second year's examination of the Institute for 1860, and we believe those solutions have been found useful by students preparing for the examination; we therefore propose to publish from time to time the questions for the subsequent years, with brief answers to such as are not taken direct from the text books, and we now accordingly submit the papers for 1861, 1862 and 1863, with such answers, Mr. Sprague having again obligingly supplied the latter. We believe it will be found that these questions have been drawn up with great care, and that there is much in them which deserves the attention, not only of the student, but of all persons interested in the theory of life assurance.—Ed. *J. I. A.*

### Questions for 1861.

1. Given—

$$\log_e 2 = \cdot693147$$
$$\log_e 3 = 1\cdot098612$$
$$\log_e 10 = \frac{1}{\cdot434294}$$

Find the values of $\log_{10} 6$, $\log_{10} 8$, $\log_{10} 12$.

To how many decimal places can the results be depended upon?

$$\textit{Ans.}\ \log_{10} 6 = \frac{\log_e 2 + \log_e 3}{\log_e 10} = 1\cdot791759 \times \cdot434294 = \cdot7781501$$

$$\log_{10} 8 = \frac{3 \times \log_e 2}{\log_e 10} = 2\cdot079441 \times \cdot434294 = \cdot9030887$$

$$\log_{10} 12 = \frac{2 \log_e 2 + \log_e 3}{\log_e 10} = 2\cdot484906 \times \cdot434294 = 1\cdot0791797$$

These results can only be trusted to six decimal places, and an error of 1 may be expected in the sixth place.

2. There are $n$ balls in a bag, one of them black and the remainder white; and $n$ persons in succession draw each one ball from the bag. Prove that they all have the same chance of drawing the black ball.

3. Give expressions for the probabilities of the various contingencies which may arise out of the occurrences of three independent events, calling the events $A$, $B$, $C$, and the probabilities $a$, $b$, $c$, respectively.

*Ans.* The chance of $ABC$ all happening $=abc$.
$AB$ happening, $C$ failing $=ab(1-c)$.
$A$ „ $BC$ „ $=a(1-b)(1-c)$.
$ABC$ all failing $=(1-a)(1-b)(1-c)$.

4. An urn contains $n+1$ balls, marked with the numbers 0, 1, 2, 3, . . . $n$. A ball is successively drawn and replaced in the urn, so that the chance of drawing any given number remains the same in each trial. What is the probability that in $h$ trials the sum of the numbers drawn will be equal to $s$?

*Ans.* The probability=the coefficient of $x^s$ in the expansion of

$$\left\{\frac{1-x^{n+1}}{(n+1)(1-x)}\right\}^h.$$

*Vide* Jones *On Annuities*, *Appx. on Probs.*, p. 12.

5. Find an expression for the number of years in which any sum of money will double itself at compound interest, and prove that it is approximately equal to $\frac{69}{I}$: $I$ being the rate per cent.

*Ans.* $n=\frac{\log_e 2}{\log_e(1+i)}=\frac{\cdot 693147}{i-\frac{i^2}{2}+\ldots}=\frac{\cdot 69}{i}$ or $\frac{69}{I}$, approximately.

6. Determine the fine which ought to be paid for renewing any number of years lapsed in a lease.

7. Describe the ordinary form of a table of mortality.

8. Given a table of mortality, show how to find (1) at what age it is most probable a person of a given age will die; (2) how many years he has an even chance of living?

9. Give some account of the more recent tables of mortality, and of the materials from which they were constructed (including the published experience of Life Assurance Companies); and point out the uses for which they are most applicable.

10. Supposing a given number of marriages contracted between males of the age of 30, and females of the age of 25, find the proportion per cent. of the original number who will survive as married couples, widowers or widows, at the end of 10 years; assuming the probability of dying within 10 years at the age of 30 to be $\frac{265}{2501}$, and at the age of $25=\frac{237}{2611}$.

*Ans.* The proportion of married couples $=\frac{2236}{2501}\times\frac{2374}{2611}$; of widowers $=\frac{2236}{2501}\times\frac{237}{2611}$; of widows $=\frac{265}{2501}\times\frac{2374}{2611}$.

11. Describe a practical method of constructing a table of the logarithms of the value of £1, receivable at the end of any number of years, from 1 to 100.

12. Explain briefly the construction and use of the columns D, N, M.

13. Prove that $M_x=vN_{x-1}-N_x$.

14. Find the annual premium for an insurance of £1 on the life of $x$ provided he die within $t$ years.

15. Find an expression for the value of a deferred annuity on two lives

$x$ and $y$, to commence in $t$ years, and to continue as long as either of the lives is in existence.

16. Find the single premium for the assurance of £1, payable on the death of A aged $x$, provided he dies before B aged $y$, or within $t$ years after the death of B.

17. Find the present value of a given sum payable on the decease of A, provided he be the first that fails of the three lives A, B, and C.

18. It is commonly thought that the value of a policy which has been only one year in force, may be found by subtracting the term premium for an insurance for one year, from the net premium for the whole of life. What correction must be applied, in order to give an accurate result?

*Ans.* The difference so found is to be multiplied by $\frac{1}{vp_x}$, or $\frac{D_x}{D_{x+1}}$.

19. Give a formula for constructing a descending scale of premiums, when the premium diminishes by equal decrements every five years, and becomes zero after 20 years.

*Ans.* The premium for the first five years

$$=\frac{M_x}{N_{x-1}-\frac{1}{4}(N_{x+4}+N_{x+9}+N_{x+14}+N_{x+19})}.$$

20. Find an expression for the value of an "endowment assurance" (*i.e.*, a policy on a life $x$ payable at age $x+n$, or previous death), after it has been $t$ years in force.

$$\textit{Ans.}\quad 1-\frac{1+{}_{\overline{n-t-1}|}a_{x+t}}{1+{}_{\overline{n-1}|}a_x} \text{ or } 1-\frac{N_{x+t-1}-N_{x+n-1}}{N_{x-1}-N_{x+n-1}}\cdot\frac{D_x}{D_{x+t}}.$$

21. An annuity is granted on the longest of three lives, A, B, and C, to be equally divided between them whilst they are all living; on the decease of either, to be equally divided between the survivors during their joint lives, and then to go to the last survivor during his life.

Give the formula for the value of A's interest therein.

22. Find the value of a reversionary annuity, to commence on the death of $x$, and continue for the remainder of the life of $y$; but to be payable for $t$ years, whether $y$ is alive or dead.

$$\textit{Ans.}\quad \left(\frac{1}{i}-a_x\right)(1-v^t)+v^t p_{y,t}(a_{y+t}-a_{x.y+t}).$$

23. Find the annual premium for a deferred annuity on a life $m$, to commence at the end of $t$ years, with the condition that if $m$ die within $t$ years, the premiums paid are to be returned with their accumulations at compound interest.

*Ans.* The annual premium

$$=\frac{N_{m+t}}{N_{m-1}-N_{m+t-1}-(1+j)M_m-(1+j)^2M_{m+1}\ldots-(1+j)^tM_{m+t-1}+\frac{1+j}{j}\{1+j)^t-1\}M_{m+t}}$$

24. If the rate at which the premiums are accumulated is the same as that at which they are calculated, show that the expression for the annual premium becomes $\frac{da_{m+t}}{(1+i)^t-1}$; $d$ being equal to $\frac{i}{1+i}$.

25. A debenture for £100, bearing interest at the rate of 6 per cent. per annum, and redeemable in 20 years, is purchased for £$107\frac{1}{2}$. Show how to determine approximately the rate of interest realized—given that at 5 per cent. the debenture would be worth £112. 9*s*. 3*d*.

*Ans.* £5. 7*s*. 7*d*. per cent.

QUESTIONS FOR 1862.

1. Show that in all systems the log. of unity is nothing.

2. Having given the log., to find the corresponding number; or, in other words, expand $a^x$ in a series proceeding by powers of $x$.

3. If the chance of any one of $n$ persons dying in the course of a year be represented by $p$, then the chance of exactly $r$ of them dying in the course of a year will be equal to

$$\frac{\lfloor n}{\lfloor r \lfloor n-r} p^r(1-p)^{n-r}.$$

4. On the suppositions in the previous question, prove that the most probable number of deaths in the year is the greatest integer contained in $(n+1)p$.

5. If two joint proprietors have an equal interest in a freehold estate worth £$p$ per annum, but one of them purchase the whole to himself by allowing the other an equivalent annuity of £$q$ for $n$ years; find the relation of $p$ to $q$.

*Ans.* $q=\frac{1}{2}\frac{p}{1-v^n}$.

6. If P represent the population of any place at a certain time, and every year the number of deaths is $\frac{1}{p}$th, and the number of births $\frac{1}{q}$th, of the whole population at the beginning of that year, required the population at the end of $n$ years from that time.

7. Given a table of annual premiums; show how to construct a corresponding table of mortality.

*Ans.* If $P_x$ be the annual premium at the age $x$, $p_x$ the chance of living a year, then $p_x=(1+i)\left(\frac{1}{P_x+d}-1\right)(P_{x+1}+d)$.

8. Give some account of the Northampton table of mortality, and point out the fundamental error in its construction.

9. The annual mortality at each age being given, show how to construct therefrom a table of the mean duration (or expectation) of life.

10. To which of the ordinary tables of premium published in a prospectus is the Carlisle table of mortality inapplicable as a basis of calculation, and why?

11. Find (by the old method) the value of an annuity on a single life, and show how the annuity at any age may be given in terms of the next higher age.

12. Explain the several columns of a single life commutation table.

13. Describe the actual construction of column D, and point out how the results may be verified.

14. Find the value of an annuity on the survivor of any number of lives—that is, to continue so long as any one of them exist.

15. Prove that if $\frac{1}{4}$ be added to the value of a life annuity of £1 payable yearly, the sum will be very nearly equal to the value of the same annuity payable half yearly.

16. Investigate a formula for the annual premium for a deferred annuity, with a return of all the premiums in the event of previous death.

17. Explain the method of calculating a table of ascending premiums, giving the proper formulæ.

What precaution must always be observed in practice with regard to the magnitude of these premiums, and how would you fix the premium to be charged during the first term of years?

*Ans.* The premium charged for the first term must not be less than the premium for a term insurance of that duration, and should be taken equal to the term premium for, say, three years older.

18. Prove that the value of £1 to be received on the decease of a person of a given age is greater than that of £1 receivable at the expiration of a number of years certain equal to the expectation of life at that age.

19. Required the present value of a sum of money to be received at the end of the year in which A dies, provided he die while B is living.

20. Give an expression for the value of a contingent reversion expectant on the decease of A, provided B be then living and C dead.

21. Find the value of a contingent life interest for the lives of A and B and the survivor, after the death of C; subject to the condition that A shall survive C.

22. Find an expression for the value of a reversionary annuity of £1 to commence on the death of a person now aged $x$, and continue payable till the expiration of $t$ years from the present time, and as much longer as a person now aged $y$ shall survive.

*Ans.* $\dfrac{1-v^t}{i} - {}_{t|}a_x + a_{y\overline{t|}} - a_{xy\overline{t|}}$.

23. Find the amount of an annuity of £1 payable quarterly in advance foı $n$ years certain, supposing that the investments made at the beginning of each year are at the rate of £$i$ per £1 per annum payable yearly, and those made in the interval are at the rate of £$j$ per £1 per annum payable quarterly.

*Ans.* $\frac{1}{4}\left\{1+i+\dfrac{4+j}{j}\left[\left(1+\dfrac{j}{4}\right)^3-1\right]\right\}\cdot\dfrac{(1+i)^n-1}{i}$.

24. If the rate of mortality in one table be throughout greater than that in another table, should you expect the values of policies obtained in the ordinary way from the first table to be greater or less than those obtained from the second? State your reasons.

*Ans.* No conclusion can be drawn from the simple fact of the rate of mortality being greater.

Prove that, under certain conditions, the values of all policies as given by two different tables of mortality will be equal.

*Ans.* This will be the case if $\dfrac{1+a_x}{1+\acute{a}_x}$ is constant, or has the same value for all values of $x$, $a_x$ and $\acute{a}_x$ being the annuities in the two tables of mortality.

25. Find the annual premium for an assurance of £1 payable at the expiration of $m+n$ years if A be then alive, or at the death of A if that

take place after the expiration of $m$ years but before the expiration of $m+n$; with the further condition, that if A die within $m$ years from the present time the premiums paid are to be returned, with the exception of the first.

*Ans.* $$\frac{M_{x+m}-M_{x+m+n}+D_{x+m+n}}{N_{x-1}-N_{x+m+n-1}-R_{x+1}+R_{x+m}+(m-1)M_{x+m}}.$$

QUESTIONS FOR 1863.

1. Explain what is meant by the common logarithm of a number; and show that if several numbers differ only in the position of the decimal point, their common logarithms have all the same decimal part.

2. Prove the principle of proportional parts in common logarithms; and show that with tables of logarithms extending to seven figures, we may expect to find the number corresponding to a given logarithm correct to seven figures, but not more.

3. From a bag containing 2 guineas, 3 sovereigns, and 4 shillings, a person draws three coins: what is the previous value of his expectation?

*Ans.* £1. 15*s.* 4*d.*

4. A halfpenny being tossed one hundred times, what is the chance that it will turn up head exactly fifty times?

If head should turn up fifty times running, is it more likely that head or tail will occur in the next trial?

*Ans.* (1) $\frac{100\times 99\times 98\ldots\times 51}{1\times 2\times 3\ldots\times 50}\cdot\frac{1}{2^{100}}$. (2) If head and tail are equally probable, no conclusion can be drawn from head turning up fifty times running; but if it is not known beforehand that head and tail are equally probable, then it is a fair conclusion that head is more likely to turn up than tail.

5. Compare the chances of throwing a single ace in one trial with two dice, and in two trials with three dice.

*Ans.* The chances are $\frac{5}{18}$ and $\frac{2975}{5184}$.

6. Find the amount at compound interest in $n$ years, of an annuity certain of £1 payable in advance.

Explain how the amount of such an annuity is to be obtained readily from the ordinary tables of annuities certain.

*Ans.* (1) $(1+i)\frac{(1+i)^n-1}{i}$. (2) The tables usually give the value of an annuity payable at the end of each year, and the value in question is found by subtracting 1 from the tabular value of an annuity for $n+1$ years.

7. The rate of interest being 5 per cent., payable half-yearly, and the income tax 7*d.* in the pound, find the sum which in two years will amount to £1,000. Given that

$$\log 9{\cdot}08534 = {\cdot}9583408$$
$$\log 1{\cdot}0242708 = {\cdot}01041479.$$

*Ans.* £908. 10*s.* 8*d.*

8. Explain what is meant by a table of mortality, and describe its most usual and convenient form.

9. If $p_{x|n}$ denote the probability that a person of the age $x$ will live for $n$ years, and $p_{x+r}$ the probability that a person of the age $x+r$ will live a year, prove that

$$p_{x|n}=p_x \cdot p_{x+1} \cdot p_{x+2} \cdot \ldots \cdot p_{x+n-1}.$$

10. Describe very briefly the methods by which the tables of mortality known as the "Northampton" and the "Experience" were formed; and the materials on which they are based.

11. Point out the chief source of error in the formation of the Northampton Table, and state what has been its effect on the table.

How does this affect the values of policies as found by the Northampton Table?

*Ans.* Dr. Price assumed that the population was stationary, neglecting the increase of the younger lives by immigration; and the table therefore represents the mortality in the first half of the table as much higher than it really is. The result is that the values of policies found by this table are generally too small.

12. What is the principal objection to the use of the Experience Table of Mortality? Is it of any practical importance?

*Ans.* That it was based on the observation of policies instead of lives, but this is probably of little consequence.

13. Prove the formulæ—

$$a_x=\frac{vl_{x+1}+v^2l_{x+2}+\ldots\ldots}{l_x}$$

$$A_x=\frac{vd_x+v^2d_{x+1}+\ldots\ldots}{l_x}$$

14. By means of the formulæ in the last question, show that

$$A_x=v-(1-v)a_x.$$

15. Describe a practical method of computing a table of annuities certain at any assumed integral rate of interest; explaining how the calculations are to be verified as they proceed.

16. Prove the formula for a contingent assurance—

$$A_{\underset{1}{\underline{x.y}}}=\frac{1}{2}\left(A_{xy}+\frac{a_{x-1.y}}{p_{x-1}}-\frac{a_{x.y-1}}{p_{y-1}}\right).$$

What supposition is made in obtaining this formula, as to the deaths which take place in the course of any year?

*Ans.* That they are distributed uniformly over the year, or take place at equal intervals therein.

17. Show that the value of £1 to be received on the failure of the *successive* lives $x$ and $y$ is $A_xA_y^{\vdots}$. When is the second life supposed to be nominated?

*Ans.* At the end of the year in which the first life dies.

18. Prove the formula for the value of a policy

$$V_{x|n}=1-(\pi_x+d)(1+a_{x+n}),$$

and point out the advantage of constructing a table of the values of policies from this formula.

19. Show that if $\pi_x$ is the annual premium for an assurance of £1 at the age $x$, then

$$\frac{\pi_x}{\pi_{x-1}}<1+\frac{1}{a_{x-1}}.$$

20. A policy is taken out, at the age $x$, for £1 payable at the age $x+m$ or previous death. When it has been $t$ years in force, the assured wishes the terms of the policy to be altered, so that the sum assured shall be payable at the age $x+m+n$, or previous death. What premium should be paid for the remainder of the extended term?

*Ans.* $\dfrac{N_{x+t-1}-N_{x+m-1}}{N_{x+t-1}-N_{x+m+n-1}} \cdot \dfrac{D_x}{N_{x-1}-N_{x+m-1}} - d.$

21. Find the annual premium for an insurance on the life $x$ of £1, together with a return of all the premiums paid; the premiums being payable for $t$ years only.

*Ans.* $\dfrac{M_x}{N_{x-1}-N_{x+t-1}-R_x+R_{x+t}}.$

22. A policy being taken out at the age $x$ for the whole of life, what portion of the $n$th annual premium is required for payment of the current claims, assuming that the rate of interest realized, and the mortality experienced, are the same as those on which the premiums are based?

*Ans.* $\left(\dfrac{1}{p_{x+n-1}}-1\right)(P_x+d)\,a_{x+n-1}$, or $v(1-p_{x+n-1})(1-V_{x|n}).$

23. A sum of money (S) is to be applied in the purchase of an annuity on three lives, $x$, $y$, $z$, such that the annual payment while all three are alive shall be A; when one has died, $\frac{4}{5}$A; and when only one survives, $\frac{2}{5}$A. Find the value of A.

*Ans.* $A=\dfrac{5S}{2(a_x+a_y+a_z)-a_{xyz}}.$

24. A debenture for £100, bearing interest at the rate of $4\frac{1}{2}$ per cent., and redeemable in 25 years, is purchased for £96. What rate of interest is obtained on the investment? N.B.—The value of the debenture, calculated at *five* per cent., is £92. 19*s*. 1*d*.

25. There are three Insurance Companies, A, B, C. In A there are 1,000 members, each insured for £100; in B there are 100 members, each insured for 1,000; and in C 10 members, each insured for £10,000. Assuming all the assured to be of the same age, and the tabular rate of mortality among persons of that age to be $1\frac{1}{2}$ per cent. per annum, show how to find the probabilities in the three Companies that claims will arise for £10,000 or upwards, in the course of a year.

*Ans.* The probabilities are—

In A, $\dfrac{|\underline{1000}}{200^{1000}}\left\{\dfrac{3^{100}\times 197^{900}}{|\underline{100}\,|\underline{900}}+\dfrac{3^{101}\times 197^{899}}{|\underline{101}\,|\underline{899}}+\dfrac{3^{102}\times 197^{898}}{|\underline{102}\,|\underline{898}}+\ldots+\dfrac{3^{1000}}{|\underline{1000}}\right\}$

In B, $\dfrac{|\underline{100}}{200^{100}}\left\{\dfrac{3^{10}\times 197^{90}}{|\underline{10}\,|\underline{90}}+\dfrac{3^{11}\times 197^{89}}{|\underline{11}\,|\underline{89}}+\dfrac{3^{12}\times 197^{88}}{|\underline{12}\,|\underline{88}}+\ldots+\dfrac{3^{100}}{|\underline{100}}\right\}$

In C, $1-\left(\dfrac{197}{200}\right)^{10}.$